

Cartan Pairs ^{*}

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Abstract

A new notion of Cartan pairs as a substitute of notion of vector fields in the noncommutative geometry is proposed. The correspondence between Cartan pairs and differential calculi is established.

1 Introduction.

A big part of the classical differential geometry on manifold Ω , see *e.g.* [13], belongs to the theory of modules over a commutative algebra $\mathcal{F}\Omega$ of smooth scalar valued functions on Ω . One defines a tangent $T\Omega$ and cotangent $T^*\Omega$ vector bundles. Their sections, vector fields and one forms respectively, constitute modules $\mathcal{X}\Omega$ and $\Lambda^1\Omega$ over $\mathcal{F}\Omega$ and are basic differential-geometric objects on Ω . Both notions (of vector fields and that of one forms) enter the game on equal rights and are mutually dual. In particular, $\Lambda^1\Omega$ can be identify with a module of $\mathcal{F}\Omega$ -linear mappings from $\mathcal{X}\Omega$ into $\mathcal{F}\Omega$ and evaluation of one form ω on a vector field X provides a $\mathcal{F}\Omega$ -bilinear pairing $\omega(X) \equiv \langle X, \omega \rangle \in \mathcal{F}\Omega$ between these modules. Also an action of vector fields on functions and an external differentiation of functions are dual each other via famous Cartan formulae

$$X(f) \equiv \langle X, df \rangle \equiv i_X df \in \mathcal{F}\Omega . \quad (1.1)$$

It appears that the Leibniz rule

$$d(fg) = (df) \cdot g + f \cdot dg \quad (1.2)$$

for an external differential $d : \mathcal{F}\Omega \rightarrow \Lambda^1\Omega$ of functions into one forms and derivation property of vector fields (the Leibniz rule for an "internal" derivation

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$$X : \mathcal{F}\Omega \rightarrow \mathcal{F}\Omega)$$

$$X(fg) = X(f)g + fX(g) \quad (1.3)$$

are related each other by (1.1). A vector field can be alternatively defined as a derivation of $\mathcal{F}\Omega$ *i.e.* as an endomorphism of $\mathcal{F}\Omega$ satisfying (1.3). Therefore, the module of vector fields bears a Lie algebra structure.

To be precise, one should distinguish between a vector field X as a smooth section of $T\Omega$ and its isomorphic image $X \in \text{Der}(\mathcal{F}\Omega) \subset \text{End}(\mathcal{F}\Omega)$, which acts on functions via (1.1). These ideas have served as a basis for an algebraic generalization of concept of vector fields, so called *Lie–Cartan pairs* [7] or *Lie pseudoalgebras* [8], see also [6] for supersymmetric generalization and [8] for overview and historical remarks. An attempt to generalize this concept to noncommutative case within the framework of braided Lie algebras [5] was performed in [9] (c.f. also [11] and [10]). The notion of Lie algebras of vector fields for quantum groups has been introduced by Woronowicz [15].

Passing to the noncommutative case, the duality between forms and vector fields fails. Unlike in the commutative case, the noncommutative differential calculus is developed mainly in the covariant approach (*i.e.* by means of differential forms). A satisfactory concept of noncommutative vector fields has not been formulated yet. The reason is that the Leibniz rule (1.2) for an external differential remains unchanged also in the noncommutative setting while that one for vector fields (1.3) has to be modified. The aim of the present note is to fill this gap. We propose a new notion of *Cartan pairs* as a substitute for a concept of vector fields. Our approach is similar to the Lie–Cartan approach but we have no analogue of Lie bracket. We explore a bimodule structure instead. A Cartan pair consist an \mathbb{K} –algebra A (\mathbb{K} being a commutative ring) and A –bimodule M with suitable action of M on A . We show that a dual object to a Cartan pair is a differential calculi on an algebra A . Our main result is that (1.1) allows to reconstruct the "external" differential if we are given an action of generalized vector fields and conversely to find out the action by means of differential. An example of such action for a given noncommutative calculus can be found in [3] (c.f. [4]).

Henceforth \mathbb{K} denotes some fixed unital and commutative ring. Algebras are unital associative \mathbb{K} –algebras and homomorphisms are assumed to be unital. All objects considered here are first of all \mathbb{K} –modules. All maps are assumed to be \mathbb{K} –linear maps.

Let M be an (A, A) –bimodule (A –bimodule in short). We shall denote by dot "." the both: left and right multiplication by elements from A . For example, by bimodule axioms, one has $(f.x).g = f.(x.g) = f.x.g$ for $f, g \in A$ and $x \in M$.

The present note has a preliminary character. The full version of it with more details and proofs will be published elsewhere.

2 Cartan pairs.

Let A be an \mathbb{K} -algebra and M an A -bimodule. By an *action* of M on A we mean a \mathbb{K} -linear mapping $\beta \in \text{Hom}_{\mathbb{K}}(M, \text{End}_{\mathbb{K}}(A))$. We shall also write $M \ni x \mapsto x^\beta \in \text{End}_{\mathbb{K}}(A)$ or $A \ni f \mapsto x^\beta(f) \in A$ to denote the action.

DEFINITION 2.1. Let \mathbb{K} be a commutative and unitary ring, and let A be an unitary \mathbb{K} -algebra. A *right Cartan pair* over \mathbb{K} and A is an A -bimodule R together with a *right action* $\rho : R \rightarrow \text{End}_{\mathbb{K}}(A)$, such that

$$(f.X)^\rho(g) = f X^\rho(g) \quad (2.1)$$

and

$$X^\rho(fg) = X^\rho(f)g + (X.f)^\rho(g) \quad (2.2)$$

Observe that in the case of commutative algebras $X.f = f.X$ (c.f. Remark 3.7 below) the formulae (2.1) and (2.2) set a generalization of the Leibniz rule (1.3) we have been looking for.

In a similar manner we define a *left Cartan pair* (L, λ) consisting a bimodule L and its *left action* $L \ni X \mapsto X^\lambda \in \text{End}_{\mathbb{K}}(A)$. Now the properties (2.1) and (2.2) must be replaced by

$$(X.g)^\lambda(f) = X^\lambda(f)g \quad (2.3)$$

$$X^\lambda(fg) = f X^\lambda(g) + (g.X)(f)^\lambda \quad (2.4)$$

DEFINITION 2.2. A left (resp. right) Cartan pair (M, A) is called a bimodule of left (resp. right) *generalized vector fields* on A if the corresponding action λ (resp. ρ) is faithful.

EXAMPLE 2.3. Let Ω be a manifold and \mathbb{K} be a field of real numbers. Take $A = \mathcal{F}\Omega$ and $M = \mathcal{X}\Omega$ together with a canonical action of vector fields on function via derivations. Since algebra is commutative, the module $\mathcal{X}\Omega$ can be considered as a bimodule with a left and right multiplication coinciding. Then $(\mathcal{X}\Omega, \mathcal{F}\Omega)$ is at the same time a left and right Cartan pair. Of course it is a bimodule of generalized vector fields.

3 Dual of bimodule.

Let R be a right A -module. Recall (see *e.g.* [1]) that R^* dual of R is defined as a collection of all right A -module maps from R into A , *i.e.* $R^* = \text{Hom}_A(R, A)$. For every ordered pair of elements $x \in R$ and $X \in R^*$, the element $X(x) \in A$, the evaluation of X on x is denoted by $\langle X, x \rangle$. R^* bears a canonical left A -module structure, therefore $\langle, \rangle : R^* \times R \rightarrow A$ defines the canonical A -bilinear form (*pairing*). Summing up the following relations hold true

$$\langle X, x + y \rangle = \langle X, x \rangle + \langle X, y \rangle \quad (3.1)$$

$$\langle X, x.f \rangle = \langle X, x \rangle f \quad (3.2)$$

$$\langle X + Y, x \rangle = \langle X, x \rangle + \langle Y, x \rangle \quad (3.3)$$

$$\langle f.X, x \rangle = f \langle X, x \rangle \quad (3.4)$$

where, $X, Y \in R$ and $f \in A$.

For a right A -modules map $\alpha \in \text{Hom}_A(R_1, R_2)$ one defines its transpose α^T as a left A -module map $\alpha^T \in \text{Hom}_A(R_2^*, R_1^*)$ by the formulae [1]

$$\langle \alpha^T(X_2), x_1 \rangle_1 = \langle X_2, \alpha(x_1) \rangle_2$$

where, $x_i \in R_i$ and $X_i \in R_i^*$, $i = 1, 2$.

Let now M be an A -bimodule and let $M^* = \text{Hom}_{(-,A)}(M, A)$ denotes its right dual, *i.e.* dual of M as a right A -module. For any element $f \in A$ left multiplication by f is a right module map $f. \in \text{Hom}_{(-,A)}(M, M)$. It is easy to check that its transpose $(f.)^T \equiv .f$ is a right multiplication in M^* and that with this multiplication M^* becomes a bimodule.

DEFINITION 3.1. The A -bimodule $M^* = \text{Hom}_{(-,A)}(M, A)$ with the canonical left module structure (3.3), (3.4) and with transpose right multiplication

$$\langle X.f, x \rangle = \langle X, f.x \rangle \quad (3.5)$$

is called a right dual of a bimodule M .

In a similar way one defines a *left dual* ${}^*M = \text{Hom}_{(A,-)}(M, A)$ of bimodule M with a canonical left and transpose right A -module structure. In this case one has

$$\langle x + y, X \rangle = \langle x, X \rangle + \langle y, X \rangle \quad (3.6)$$

$$\langle f.x, X \rangle = f \langle x, X \rangle \quad (3.7)$$

$$\langle x, X + Y \rangle = \langle x, X \rangle + \langle x, Y \rangle \quad (3.8)$$

$$\langle x, X.f \rangle = \langle x, X \rangle f \quad (3.9)$$

$$\langle x.f, X \rangle = \langle x, f.X \rangle \quad (3.10)$$

It is interesting to compare a left dual of a right dual of a bimodule M with M .

PROPOSITION 3.2. *There is a canonical A -bimodule map from M into ${}^*(M^*)$ (resp. $({}^*M)^*$) $x \mapsto \tilde{x}$ given by the formulae*

$$\langle X, \tilde{x} \rangle = \langle X, x \rangle \quad (\text{resp. } \langle \tilde{x}, X \rangle = \langle x, X \rangle) \quad (3.11)$$

In general, it is neither injective nor surjective.

DEFINITION 3.3. A bimodule M is called right (resp. left) reflexive if ${}^*(M^*) \equiv M$ (resp. $({}^*M)^* \equiv M$) *i.e.* when the corresponding canonical map (3.11) is a bimodule isomorphism.

DEFINITION 3.4. An A -bimodule M is called a right (resp. left) free A -bimodule if it is so as a right (resp. left) A -module.

DEFINITION 3.5. An A -bimodule M is called a right (resp. left) finitely generated bimodule if it is so as a right (resp. left) A -module.

LEMMA 3.6. *Let M be a right (resp. left) free and finitely generated A -bimodule. Then a right (resp. left) dual of M is a left (resp. right) free finitely generated bimodule. Moreover, M is a right (resp. left) reflexive.*

REMARK 3.7. Assume that A is commutative. Each A -module becomes automatically an A -bimodule with the same left and right multiplication. In this case the three notions of dual, namely: left and right dual of bimodule and dual of module over commutative algebra, coincide. Of course, the module of the classical vector fields $\mathcal{F}\Omega$ over manifold Ω is reflexive (c.f. Example 2.3).

4 Main results.

It appears that differential calculi investigated recently by many authors in the context of quantum groups and noncommutative geometry are nothing but derivations of an algebra with values in a bimodule (c.f. [2, 3, 4, 12, 14]). Recall that a \mathbb{k} -derivation d of A to M , $d \in \text{Der}_{\mathbb{k}}(A, M)$, is a \mathbb{k} -linear mapping from A into M such that the Leibniz rule (1.2) is satisfied. The pair (M, d) is said to be *first order calculus* or *first order differential* on an algebra A with values in an A -bimodule M or shortly *M -valued calculus* on A . Each \mathbb{k} -derivation vanishes on scalars from \mathbb{k} .

Let now (M, d) be a calculus on an algebra A . The differential d and formulae (1.1) defines an action of the right dual M^* on A . This action

$$A \ni f \mapsto X^\partial(f) \equiv \langle X, df \rangle \quad (4.1)$$

will be called a *right partial derivatives* along the element $X \in M^*$ with respect to the calculus (M, d) . One uses X^∂ instead of more traditional notation $\partial/\partial X$. It can be checked that this action satisfies axioms of right Cartan pair. Therefore, to each differential calculus (M, d) on A we can associate a unique right Cartan pair of right partial derivatives (M^*, ∂) of (M, d) . The converse statement is also true: to each right Cartan pair (R, ρ) one can associate a unique differential calculus $({}^*R, d_\rho)$ where, $d_\rho : A \rightarrow {}^*R$ is defined by formulae (4.2) below. Thus we have

MAIN THEOREM. *Let (M, d) be a calculus on A . Then M^* together with an action (4.1), via the right partial derivatives, becomes a a right Cartan pair (M^*, ∂) on A . Moreover, if the module M of one forms is spanned by differential (i.e. $M = A.dA$) then the action ∂ is faithful.*

Conversely, let (R, ρ) be a right Cartan pair on A . Then the formulae

$$\langle X, d_\rho f \rangle = X^\rho(f) \quad (4.2)$$

*for each $X \in R$, determines $d_\rho f$ as an element of a left dual *R of the bimodule R . The mapping $d_\rho : A \rightarrow {}^*R$ defines an *R -valued calculus $({}^*R, d_\rho)$ on A .*

In a case of a right reflexive bimodule $M = {}^*(M^*)$ one has $d = d_\partial$ and $\rho = \partial_\rho$.

In a similar way an action

$$A \ni f \mapsto \partial X(f) \equiv \langle df, X \rangle \quad (4.3)$$

determines a left Cartan pair structure on *M (left partial derivatives).

Therefore to each differential calculus one can canonically associate a right (resp. left) Cartan pair of partial derivatives. Conversely, for each left (resp. right) Cartan pair there exists an associated differential calculus on an algebra A . In a case of reflexive bimodule a successive application of above canonical constructions give rise to the initial object.

An application of Cartan pairs in the theory of noncommutative vector bundles and connections (c.f. [4]) will be investigated elsewhere.

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